Restricting ourselves in our further discussion of the curve to one quadrant, we find:

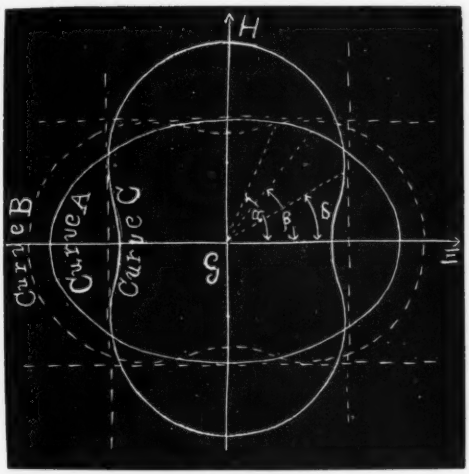


Fig. 5.

For $0 \leqq \psi < \beta$; every tangent lies outside the curve,—it has positive curvature.

For $\psi = \beta$, the curve has a double tangent, touching at β and $\pi - \beta$.

For $\beta < \psi < a$; every tangent intersects the curve,—it has positive curvature.

For $a < \psi < \frac{1}{2}\pi$; every tangent intersects the curve,—it has negative curvature.

By applying rows No. II and IV of our general theorems, we obtain:

$0 \leqq r \sin \psi \leqq r \sin \beta$	$E(x, y; \cos\theta, \sin\theta; \cos\bar{\theta}, \sin\bar{\theta}) < 0$ for $0 \leqq \bar{\theta} \leqq 2\pi$, except $\theta = \bar{\theta}$. $F_1 > 0$.	(x, y, ψ) is an element of strong minimum.
$r \sin \psi = r \sin \beta$	$F_{x'}(x, y, \cos\psi, \sin\psi) = F_{x'}(x, y, \cos(r-\psi), \sin(\pi-\psi))$. $F_{y'}(x, y, \cos\psi, \sin x) = F_{y'}(x, y, \cos(r-\psi), \sin(r-\psi))$.	(x, y, ψ) and $(x, y, r-\psi)$ are elements of discontinuous solutions.
$r \sin \beta < r \sin \psi \leqq a$	$E(x, y; \cos\theta, \sin\theta; \cos\bar{\theta}, \sin\bar{\theta})$ is of changing sign for $0 \leqq \bar{\theta} \leqq 2\pi$. $F_1 > 0$.	(x, y, ψ) is an element of weak minimum.
$a < r \sin \psi < r$	E as above. $F_1 < 0$.	(x, y, ψ) is an element of weak maximum.

Denoting row $r \sin \psi$ by d, and rembering that it represents the distance from the origin to a straight line, we observe that if $r \sin \psi$ has a certain value for one element, it will have the same value along the entire line, to which the element belongs.

We find then, that straight line segments, which lie outside C_a are *weak maxima*, which agrees with a previous result (see I on page 125).

The elements, which belong to lines intersecting C_a, belong to weak minima, if

$$d > \frac{1}{1+r^2}\sqrt{[(r^2+5)(3-r^2)]},$$

to strong minima, if

$$d < \frac{1}{1+r^2}\sqrt{[(r^2+5)(3-r^2)]}.$$

Solving the equation

$$d = \frac{1}{1+r^2}\sqrt{[(r^2+5)(3-r^2)]}$$

for r, we obtain

$$r = \sqrt{\sqrt{\frac{16}{d^2+1}} - 1}, \tag{16}$$

which enables us to determine on every line, intersecting C_a, a point which separates the segments which are strong minima, from those which are weak minima. Constructing this point for all extremals which are parallel to the X-axis, we obtain the curve, denoted by C_r in Figs. 4 and 6. For other extremals, an analogous curve can be obtained by revolving C_r until the line PP' becomes parallel to the given extremal.*

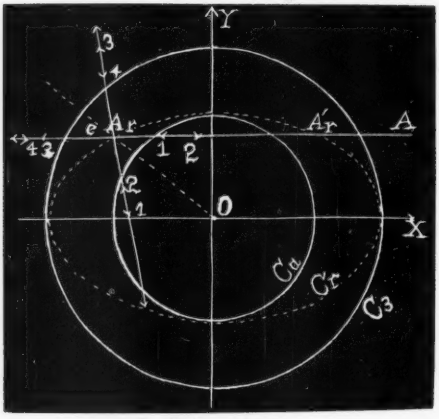


Fig. 6.†

We see further, that since (x, y, ψ) is an element of discontinuous solution, every point in this region is a corner of four discontinuous solutions, the branches being in the directions β, $\pi-\beta$, $\pi+\beta$, and $2\pi-\beta$, as indicated in Fig. 6, where 1 and 2, as well as 3 and 4 must be considered as distinct (compare A. Kneser, *Lehrbuch der Variationsrechnung*, p. 9).

From the definition of C_r follows finally, that the intersection of C_r with a line parallel to X-axis (or of an analogous curve with any other line) determine upon that line two points, which are each corners of four discontinuous solutions, of which the given line furnishes one

*For practical purposes, one would make a cardboard sample of C_r and put it in any required position to determine the points corresponding to A_r and A'_r (Fig. 6).

†ERRATUM.—In the diagram, 4 and 3 on the oblique line through A_r should be interchanged. ED. F.

of the branches. The other branches are obtained by reflecting the line on the radius vector of the point. In Fig. 6, where A is the given line, A_r and A'_r are the points referred to, while the other branches for the former point are formed by BB'.

We have now the following result:

III. *A straight line not intersecting C_a, is a weak maximum within C_3. A straight line intersecting C_a is a strong minimum between C_a and C_r, a weak minimum between C_r and C_3.*

C_r determines upon every line intersecting C_a a corner of a discontinuous solution.

Every point between C_a and C_3 can be a corner of a discontinuous solution.

§ 7. *Case III.* $\sqrt{3}<r<\sqrt{15}$.

In this case we find:

$$F(r, \psi)<0, \text{ for } 0\leqq\psi<\gamma,$$
$$F(r, \psi)=0, \text{ for } \psi=\gamma,$$
$$F(r, \psi)>0, \text{ for } \gamma<\psi\leqq\tfrac{1}{2}\pi,$$

where γ is defined by

$$\frac{\sqrt{(1+r^2\sin^2\gamma)}}{1+r^2}-\tfrac{1}{4}=0,\ 0\leqq\gamma\leqq\tfrac{1}{2}\pi;\ i.\ e.,\ \sin^2\gamma=\frac{(r^2+5)\ (r^2-3)}{16r^2}. \qquad (17)$$

$F_1(r, \psi)$ as in § 6.

Application of theorems A and B shows, that in this case, the Indicatrix has negative curvature in the interval $0\leqq\psi<\gamma$, and has a discontinuity at $\psi=\gamma$. It has further positive curvature from $\psi=\gamma$ to $\psi=\alpha$, from where on, it continues with negative curvature until $\psi=\frac{1}{2}\pi+\alpha$.* The further course of the curve is determined by its symmetry properties.

From equation (13) follows again, that the curve crosses at right angles at $\psi=0$, and at $\psi=\frac{1}{2}\pi$; further $\tan\bar{\psi}=0$, for $\psi=\beta$, *i. e.*, the curve is tangent to a line parallel to the radius vector at its point of discontinuity. To determine the asymptote, we compare the curve at $\psi=\beta$ and the straight line $\psi=\beta$. Denoting the rectilinear co-ordinates of the curve by ξ and η, and those of the straight line by $\bar{\xi}$ and $\bar{\eta}$, we find for a given $\xi=\bar{\xi}$:

$$\eta=\rho\sin\psi=\frac{\sin\psi}{F},\quad \bar{\eta}=\tan\beta.\bar{\xi}=\tan\beta.\xi=\frac{\tan\beta\cos\psi}{F}.$$

*This discussion is made under the assumption that $\gamma<\alpha$; if $\alpha<\gamma$, the curve is of a somewhat different nature. This does not affect the conclusions derived in this consideration however, for which reason the cases are not distinguished. Comparing equations (12) and (17), we find that

$$\alpha\gtreqless\gamma, \text{ for } r\lesseqgtr\sqrt{(4^4-1)}.$$

Consequently:

$$\eta - \bar{\eta} = \frac{\sin(\psi - \beta)}{\cos^2 \mathrm{F}},$$

$$\text{and} \quad \lim_{\psi=\beta} (\eta - \bar{\eta}) = \frac{16(r^2+1)}{(15-r^2)\sqrt{[(r^2+5)(r^2-3)]}}, \tag{18}$$

which is real and positive for all points between C_3 and C_{15}. It follows, that the asymptote is parallel to the line $\psi=\beta$, and intersects the Y-axis at the point:

$$\eta = \frac{16(r^2+1)}{(15-r^2)\sqrt{[(r^2+5)(r^2-3)]}}.$$

We conclude now immediately, that every tangent intersects the curve, and that no double tangents are possible, from which by theorems *C*, *D* and IV, V, and by the reasoning given in § 6 for the connection between extremal elements and extremals, we derive the following results:

IV. *A straight line, not intersecting C_a is a weak maximum between C_3 and C_{15}.*

A straight line, intersecting C_a is a weak minimum between C_3 and C_{15}.

No point in the region between C_3 and C_{15} can be a corner of a discontinuous solution.

§ 8. *Case IV.* $\sqrt{15} < r$.

We have here finally:

$\mathrm{F}(r, \psi) < 0$, $0 \leqq \psi \leqq 2\pi$.

$\mathrm{F}_1(r, \psi)$ as in § 6.

We find here, by application of theorems *A* and *B*, that the Indicatrix is closed around the origin, has negative curvature for $0 \leqq \psi < a$, and positive curvature for $a < \psi \leqq \frac{1}{2}\pi$, again having double symmetry. As in case II, the curve crosses at right angles for $\psi=0$ and $\psi=\frac{1}{2}\pi$, and we want to determine the value of ψ, different from 0 for which it has a vertical tangent. Denoting that value by δ, we find by transforming to rectangular coordinates (14), by a method similar to the one used for the determination of β:

$$\sin^2 \delta = \frac{15}{r^2}, \tag{19}$$

which furnishes a real value of δ for any point outside C_{15} (see Fig. 5, curve *C*).

The case bears very much likeness to the one discussed in § 6, for which reason, we conclude briefly, using theorems *C*, *D*, IV, and V:

$0 \leqq r \sin.\psi \leqq a$	(x, y, ψ) is an element of weak minimum.
$a < r \sin.\psi < \sqrt{15}$	(x, y, ψ) is an element of weak maximum.
$r \sin.\psi = \sqrt{15}$	(x, y, ψ) and $(x, y, 2\pi - \psi)$ are elements of discontinuous solutions.
$\sqrt{15} < r \sin.\psi$	(x, y, ψ) is an element of strong maximum.

We notice, that the branches of the discontinuous solutions are determined by the tangents to C_{15}, so that, although every point of the region is a corner of four discontinuous solutions (see Fig. 7), we can not determine on *every* line a corner of a discontinuous solution of which the given line is a branch, as could be done in case II, whereas on some lines we can determine an infinitude of corners, which was impossible in II.

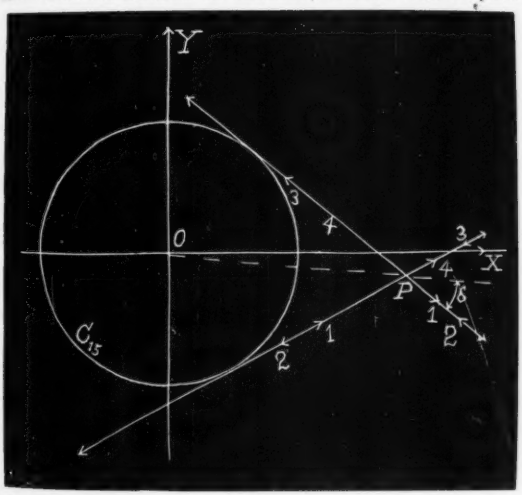


Fig. 7.*

We have here the following result:

V. A straight line intersecting C_a is a weak minimum outside C_{15}. A straight line, not intersecting C_a, but intersecting C_{15}, is a weak maximum outside C_{15}. A straight line, lying entirely outside C_{15}, is a strong maximum.

Every point outside C_{15} is a corner of four discontinuous solutions, which can be determined by drawiag the tangents to C_{15}.

§ 9. *Limiting cases and final results.*

We consider in conclusion:

*Erratum.—δ should be between the dotted line and the lower oblique line through *P*.

(a) $r=a$. We have then:

$F(r, \psi)>0,\ 0\leqq\psi\leqq 2\pi.$

$F_1(r, \psi)>0,\ 0\leqq\psi\leqq 2\pi$, excepting $\psi=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$.

The Indicatrix is of the same nature as for case I. Consequently, any extremal-element on C_a is an element of a *strong minimum*, while no discontinuous solutions can have a corner on C_a.

(b) $r=1/3$.

$F(r, \psi)=0$ for $\psi=0$.

$F(r, \psi)>0$ for $0<\psi\leqq\frac{1}{2}\pi$.

$F_1(r, \psi)$ as in § 6.

The Indicatrix is of the nature described in § 7, and we find from equation (18):

$$\lim_{\psi\doteq i\beta}(\eta-\bar{\eta})=\infty,$$

from which follows, that the curve has a parabolic character, so that we conclude: Any extremal-element in C_3 is an element of a *weak minimum*, if $\psi<a$; it is an element of a *weak maximum*, if $\psi>a$.

(c) $r=1/15$.

$F(r, \psi)>0$, for $0\leqq\psi<\frac{1}{2}\pi$.

$F(r, \psi)=0$, for $\psi=\frac{1}{2}\pi$.

$F_1(r, \psi)$ as in § 6.

We find again from equation (19):

$$\lim_{\psi=i\beta}(\eta-\bar{\eta})=\infty,$$

from which we derive the same results as in sub (b).

We combine now the results I, II, III, IV, and V, obtained §§ 3, 5, 6, 7, 8 for lines parallel to the X-axis (see Fig. 4):

1. *D is a strong maximum along its entire length and allows four discontinuous solutions at every point.*

2. *C is a weak maximum along its entire length; every point outside of the interval $C'_{15}C_{15}$ is a corner of four discontinuous solutions.*

3. *B is a weak maximum along its entire length; every point on the segment B'_3B_3 is a corner of four discontinuous solutions.*

4. *A is a weak minimum outside the segment* A'_rA_r; it is a strong minimum from A_a up to A_r and from A'_a up to A'_r; it is a hyperstrong minimum on the segment A'_aA_a; all points outside the interval $A'_{15}A_{15}$ and all points on the segments $A'_3A'_a$ and A_3A_a are the corners of four discontinuous solutions; A_r in particular is the corner of the discontinuous solutions, of which one branch lies along A.*

*The segment AB excludes the points A and B; the interval AB includes the points A and B.

ON THE TRISECTION OF AN ANGLE.

By E. E. WHITE, M. E., Harvard University.

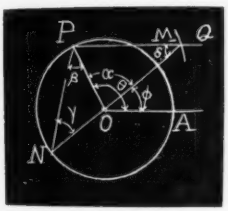
The two following approximate constructions for the trisection of an angle may be interesting. They are, of course, essentially methods of trial and error, since it is only after placing the straight edge in position, passing through the given point, that one can tell whether the position is that desired. The straight-edge must be continually shifted till all the conditions are fulfilled, as nearly as can be told by eye, and mathematically speaking the methods are therefore only approximate.

While it is well known that an angle cannot be trisected by *strict* geometrical construction, nevertheless there are many interesting and practically useful approximate constructions, of which the following are examples. The principle of neither construction is new, though the author devised the first method independently. Four years ago Mr. John J. Quinn made two linkages which involve these constructions, and the linkages in turn depend upon the limaçon. Further, the construction using a graduated scale is an application of the method devised by Archimedes, known as the "Method of Insertions."

CONSTRUCTION I.

Given angle θ, to trisect the angle.

With the vertex, O, as a center, describe a circle of any radius, intersecting the sides of the angle at P and A. Through P draw the line PQ parallel to OA by the usual method. Pass a straight-edge through O, and adjust the position of the straight-edge so that its intersection N with the circle O, and its intersection M with the line PQ shall be equidistant from P (as tested by compasses). Draw the line NOM, which will trisect the angle.



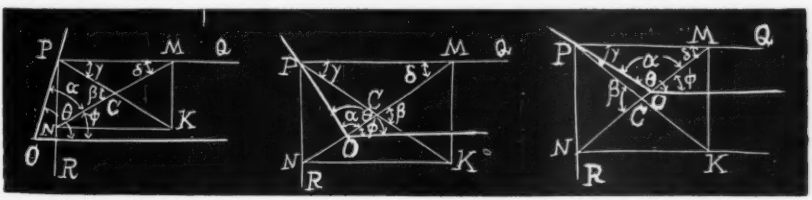
Proof. Draw the line PN.

Then $\theta=a+\phi=\beta+\gamma+\phi=2\gamma+\phi=2\delta+\phi=2\phi+\phi=3\phi$.

CONSTRUCTION II.

Given angle θ, to trisect the angle.

Through any point P on one side of the angle draw PQ parallel and



PR perpendicular to the opposite side of the angle. Mark off a distance equal to $2\overline{OP}$ on a straight-edge passing through *O*, and adjust the straight-edge so that one mark falls on line *PQ* at *M*, and the other on line *PR* at *N*. Draw the line *NOM*, which will trisect the angle.

Proof: Complete the rectangle *PMKN*, and draw the other diagonal *PK*. Then $\overline{OP}=\frac{1}{2}\overline{MN}=\frac{1}{2}\overline{PK}=\overline{PC}=\overline{CM}$.

Hence, $\theta=\alpha+\phi=\beta+\phi=\gamma+\delta+\phi=2\delta+\phi=2\phi+\phi=3\phi$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

283. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Solve $w+x+y+z=4a$, $w^2+x^2+y^2+z^2=4a^2+4b^2$, $w^3+x^3+y^3+z^3=4a^3+12ab^2$, $w^4+x^4+y^4+z^4=4a^4+4b^4+4c^4+24a^2b^2$.

Solution by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

The following method applies equally well to the corresponding equations with arbitrary constant terms. We are given s_1, s_2, s_3, s_4, where s_n is the sum of the nth powers of w, x, y, z. Hence the latter are, by Newton's identities, the roots of the following quartic:

$$\xi^4-4a\xi^3+(6a^2-2b^2)\xi^2+(4ab^2-4a^3)\xi+a^4+b^4-2a^2b^2-c^4=0.$$

To obtain the reduced quartic, set $\xi=\eta+a$. Then

$$\eta^4-2b^2\eta^2+b^4-c^4=0, \quad (\eta^2-b^2)^2=c^4.$$

Hence, the 24 sets of solutions are given by the arrangements of $a\pm\sqrt{(b^2\pm c^2)}$.

Similarly solved by G. B. M. Zerr and J. Scheffer.

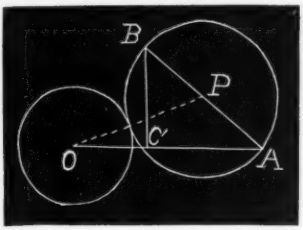
GEOMETRY.

312. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania, Philadelphia, Pa.

A variable circle passes through a fixed point and is tangent to a given circle. If a diameter of the first circle passes through the fixed point, find the locus of its other extremity.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let O be the center of the given circle, radius$=r$, A the given point, $OA=b$, $OC=x$, $BC=y$. Denoting the radius of the variable circle P by ρ, then



$$(r+\rho)^2=\tfrac{1}{2}y^2+\tfrac{1}{4}(b+x)^2 \text{ and } 4\rho^2=y^2+(b-x)^2.$$

Eliminating ρ, we obtain the equation of the locus of B,

$$y^2=\frac{b^2-r^2}{r^2}(x^2-r^2),$$

a hyperbola, with semi-conjugate axes r and $\frac{1}{r}(b^2-r^2)$. Consequently, A will be a focus.

Also solved by J. S. Brown.

313. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that an algebraic curve of odd degree which is symmetrical with respect to a center has the center on the curve.

I. Solution by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

By a transformation of coordinates we may assume that the center is at the origin. If the resulting equation of odd degree is $f(x, y)=0$, then must $f(-x, -y)=0$ for all values x, y satisfying $f(x, y)=0$. But the terms of highest degree in $f(x, y)$ change sign when x and y are changed in sign. Hence $f(-x, -y)\equiv -f(x, y)$, so that the terms of even degree vanish. In particular, the constant term is zero, so that the origin lies on the curve.

II. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

If in the equation of any locus we put $-x$, $-y$ for x, y, respectively, we get a locus symmetrical to the given locus with respect to the origin. In case the locus is symmetrical to itself, the latter equation will represent the same locus and differ from the original equation only by a constant multiplier. Let

$$0=u_0+u_1+u_2+u_3+\ldots+u_r+\ldots$$

be the equation of a locus symmetrical with respect to the origin, u_n denoting the terms of degree n, and r being an *odd* integer. Another equation to this locus, obtained by putting $-x$, $-y$ for x, y, respectively, and changing the sign of each term, is

$$0=-u_0+u_1-u_2+u_3-\ldots+u_r-\ldots$$

Since each equation contains the term u_r, the constant multiplier is unity.

Therefore

$$0=u_0=u_2=u_4=\ldots$$

That is, *if a locus symmetrical with respect to the origin contains one term of odd degree, it contains no absolute term and no terms of even degree.* A further expansion of the problem leads to the theorem that *if a locus symmetrical with respect to the origin contains one term of even degree, all the terms are of even degree.*

CALCULUS.

239. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

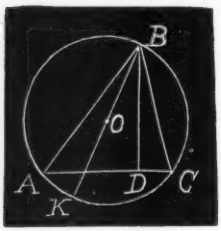
Of all triangles inscribed in a circle, find that which has the greatest perimeter.

Solution by C. N. SCHMALL, 89 Columbia Street, New York City, and REV. J. H. MEYER, S. J., Augusta, Ga.

Let ABC be the required triangle; O, the center of the given circle; $BK,=2r$, the diameter of the circle; $\angle BAC=\phi$, and $\angle BCA=\psi$. Then the perimeter, $p,=AB+BC+AC=$maximum... (1).

But $AB.BC=2r.BD$, whence $BC=2r\dfrac{BD}{AB}=2r\sin\phi$.

Also $\dfrac{AB}{BC}=\dfrac{\sin\psi}{\sin\phi}$, whence $AB=BC\dfrac{\sin\psi}{\sin\phi}$, which, from the previous equation, $=2r\sin\psi$. We also have, from the law of signs,



$$AC=\frac{\sin(\phi+\psi)}{\sin\phi}BC=2r\sin(\phi+\psi).$$

Substituting in (1), we have

$$p=2r[\sin\phi+\sin\psi+\sin(\phi+\psi)]\ldots(2).$$

Hence, for a maximum or minimum,

$$\frac{\partial p}{\partial\phi}=\cos\phi+\cos(\phi+\psi)=0, \text{ and } \frac{\partial p}{\partial\psi}=\cos\psi+\cos(\phi+\psi)=0.$$

Hence, $\cos\phi=\cos\psi$ and $\phi=\psi$. Since $\cos\phi+\cos(\phi+\psi)=0$, we have, by substitution, $\cos\phi+\cos 2\phi=0$, or $\cos\phi+2\cos^2\phi-1=0$, whence, $\cos\phi=\frac{1}{2}$, and therefore $\phi=60^\circ=\psi$.

Hence, the triangle is equilateral.

It is easy to show that $\dfrac{\partial^2 p}{\partial\phi^2}\cdot\dfrac{\partial^2 p}{\partial\psi^2}>\dfrac{\partial^2 p}{\partial\phi\,\partial\psi}$, and that, therefore, the triangle is a maximum.

Also solved by A. F. Carpenter, G. B. M. Zerr, A. H. Holmes, J. Scheffer, and G. W. Greenwood.

Professors Zerr, Greenwood, and Scheffer, and Mr. Holmes denoted the angles at the center subtended by the sides, by 2θ, 2ϕ, and 2ψ, and showed that these angles are equal to 120° each. Professor Carpenter showed that the inscribed triangle with one side constant and of maximum perimeter is isosceles, and then showed that of all isosceles triangles inscribed in a circle, the equilateral has the maximum perimeter.

240. Proposed by L. MORDELL, Philadelphia, Pa.

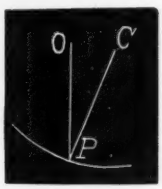
Show that the osculating conic of the catenary $y=c\cosh\frac{x}{c}$ at the point for which $y=\frac{c\sqrt{10}}{2}$ is a parabola.

I. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let O be the center of curvature of the point considered, and C the center of the conic of closed contact, then (*vide* Joseph Edwards' *Differential Calculus*, where the problem is proposed as an exercise):

$$\frac{\cos\phi}{R}=\frac{1}{\rho}-\frac{\partial\phi}{\partial s},\quad\text{and}\quad\phi=\tan^{-1}\tfrac{1}{3}\cdot\frac{\partial\rho}{\partial s},$$

where $\phi=\angle OPC$, $OP=\rho=$radius of curvature, s an arc of the given curve, and $PC=R$. From



$$y=\frac{c}{2}(e^{x/c}+e^{-(x/c)}),\ \text{we find}\ \frac{\partial y}{\partial x}=\tfrac{1}{2}(e^{x/c}-e^{-(x/c)}),$$

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{2c}(e^{x/c}+e^{-(x/c)},\quad\frac{\partial s}{\partial x}=\tfrac{1}{2}(e^{x/c}+e^{-(x/c)}).$$

$$\therefore\rho=\left[\left(\frac{\partial s}{\partial x}\right)^3\Big/\left(\frac{\partial^2 y}{\partial x^2}\right)\right]=\frac{c}{4}(e^{x/c}+e^{-(x/c)})^2.\quad\therefore\frac{\partial\rho}{\partial s}=\frac{\partial\rho}{\partial x}\cdot\frac{\partial x}{\partial s}=e^{x/c}-e^{-(x/c)},$$

therefore, $\phi=\tan^{-1}\frac{e^{x/c}-e^{-(x/c)}}{3}$, or, $e^{x/c}-e^{-(x/c)}=3\tan\phi$.

$$\therefore\frac{\partial\phi}{\partial x}=\frac{(e^{x/c}+e^{-(x/c)})\cos^2\phi}{3c}.\quad\therefore\frac{\partial\phi}{\partial s}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial s}=\frac{2}{3c}\cos^2\phi.$$

For the value $y=\frac{c}{2}\sqrt{10}$, we find $\tan\phi=\frac{1}{3}\sqrt{6}$, $\cos^2\phi=\frac{3}{5}$, $\rho=\frac{5c}{2}$, $\frac{\partial\phi}{\partial s}=\frac{2}{5c}$; substituting in $\frac{\cos\phi}{R}=\frac{1}{\rho}-\frac{\partial\phi}{\partial s}$, we find $\frac{\sqrt{\frac{3}{5}}}{R}=\frac{2}{\rho c}-\frac{2}{\rho c}=0;\quad\therefore R=\infty$,

and since the conic, the center of which is at an infinite distance, is a parabola, the assertion is proved.

II. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

In the catenary, $y=c\sec\psi$, we have $s=c\tan\psi$ and $\rho=c\sec^2\psi$.

If a conic whose center is C osculates a curve at a point P, the center of curvature for the curve at that point being O, we have

$$\tan\phi=\tfrac{1}{3}.\frac{d\rho}{ds}, \text{ and } \frac{\cos\phi}{PC}=\frac{1}{\rho}-\frac{d\phi}{ds}, \text{ where } \phi=\angle OPC.$$

[Edwards' *Differential Calculus*, p. 356, 3d edition.]

$$\rho=c(1+\tan^2\psi)=c+\frac{s^2}{c}, \quad \frac{d\rho}{ds}=2\tan\psi.$$

$$\therefore \tan\phi=\tfrac{2}{3}\tan\psi; \ (1+\tan^2\phi)d\phi=\tfrac{2}{3}(1+\tan^2\psi)d\psi.$$

$$\therefore \frac{\cos\phi}{PC}=\frac{1}{\rho}-\frac{d\phi}{d\psi}\frac{1}{\rho}=\frac{1}{\rho}\left(1-\frac{6(1+\tan^2\psi)}{9+4\tan^2\psi}\right)=\frac{3-2\tan^2\psi}{\rho(9+4\tan^2\psi}=0,$$

when $y=c\sec\psi=\dfrac{c\sqrt{10}}{2}$.

Hence (since $\cos\phi\neq 0$), PC is infinite, and the conic is a parabola.

This question was set at Oxford in 1889.

Also solved by G. B. M. Zerr.

MECHANICS.

200. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

An elastic string whose weight is W is laid over the top of an inclined plane so as to remain at rest. Determine how much the string will be elongated, knowing, M=modulus of elasticity, L=normal length of string, and ϕ=inclination of the plane.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let M=mass of unit length of the unstretched string. The equation of equilibrium is $dT+Mg\sin\varphi ds=0$. But $MgL=W$.

$$\therefore T+\frac{W\sin\varphi s}{L}=\text{constant}. \quad \text{Now when } s=0, \ T=W\sin\varphi.$$

$$\therefore T=W\sin\varphi(1-\frac{s}{L}). \quad \text{Also } dl=(1+\frac{T}{M})ds=[1+W\sin\varphi(\frac{1}{M}-\frac{s}{LM})]ds.$$

$$l=\int_0^L[1+W\sin\varphi(\frac{1}{M}-\frac{s}{LM})]ds=L[1+W\sin\varphi(\frac{1}{M}-\frac{1}{2M})]$$

$$=L[1+\frac{W\sin\varphi}{2M}]. \quad \therefore \frac{WL\sin\varphi}{2M} \text{ is the elongation.}$$

Also solved by the Proposer.

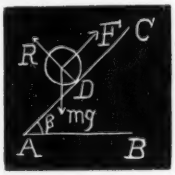
201. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

ABC is an inclined plane, perfectly rough, length $AC=l$. The time for a sphere to roll down when AB is base is to the time for a cylinder to roll down when BC is base as m is to n. Find AB and BC.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let O be the center of any rolling body of mass m; then the three forces that will act on it are its weight mg vertically downward, the resistance R on the inclined plane AC, and the friction F acting up the plane.

Denoting CD by s, we have therefore $m(\partial^2 s/dt^2)=mg\sin\beta-F$; and if we denote by θ the angular velocity, reducing the mass to the center O, we have $m\rho^2(\partial^2\theta/\partial t^2)=Fr$, r being equal to DO, and ρ radius of gyration. Since there is no sliding, the plane being perfectly rough, we have $s=r\theta$. Eliminating F, we have $\partial^2 s/dt^2=[r^2/(r^2+\rho^2)]g\sin\beta$, and integrating



$$s=\frac{r^2}{r^2+\rho^2}g\sin\beta\, t^2.$$

For the sphere $\rho^2=\frac{2}{5}r^2$, and for the cylinder $\rho^2=\frac{1}{2}r^2$; therefore, by the condition of the problem, for the sphere

$$l=\frac{r^2}{r^2+\frac{2}{5}r^2}g\,.\frac{BC}{l}t^2, \text{ whence } t^2=\frac{14l^2}{5g.\overline{BC}}$$

and for the cylinder, $l=\frac{r^2}{r^2+\frac{1}{2}r^2}g\,.\frac{AB}{l}t^2$, whence $t^2=\frac{3l^2}{g.\overline{AB}}$; $\therefore \frac{14}{5\overline{BC}}:\frac{3}{\overline{AB}}=m^2$ $:n^2$, and combining this with $\overline{AB}^2+\overline{CB}^2=l^2$, we get

$$AB=\frac{14n^2l}{\sqrt{(225m^4+196n^4)}}, \quad CB=\frac{15m^2l}{\sqrt{(225m^4+196n^4)}}.$$

Also solved by J. Edward Sanders, and the Proposer.

AVERAGE AND PROBABILITY.

185. Proposed by R. D. CARMICHAEL, Anniston, Ala.

If a line l is divided into n parts by $n-1$ points taken at random on it, what is the mean value of the pth power of one of the parts taken at random?

Solution by HENRY HEATON, Belfield, N. D.

By making the line the circumference of a circle it will be seen that each part has exactly the same chance, so that the average of one part is the same as that of any other. Hence it will be sufficient to find the average length of any one part.

Let $x_1, x_2, \ldots, x_{n-1}$ be the distance of the different points from one extremity of the line. Then the average length of $(l-x_{n-1})^p$ is

$$A=\frac{\int_0^l \int_0^{x_{n-1}} \cdots \int_0^{x_2} (l-x_{n-1})^p dx_{n-1} dx_{n-2} \cdots dx_1}{\int_0^l \int_0^{x_{n-1}} \cdots \int_0^{x_2} dx_{n-1} dx_{n-2} \ldots dx_1}$$

$$=\frac{\int_0^l (l-x_{n-1})^p x_{n-1}^{n-2} dx_{n-1}}{\int_0^l x_{n-1}^{n-2} dx_{n-1}} = (n-1) l^p B[(p+1),\ (n-1)]$$

$$= \frac{(n-1) l^p\, [\Gamma(p+1),\ \Gamma(n-1)]}{\Gamma(p+n)} = \frac{l^p\, [\Gamma(p+1),\ \Gamma(n)]}{\Gamma(p+n)}$$

or, the average length of $x_1{}^p$ is

$$A=\frac{\int_0^l \int_0^{x_{n-1}} \cdots \int_0^{x_2} dx_{n-1}\, dx_{n-2} \ldots x_1{}^p dx_1}{\int_0^l \int_0^{x_{n-1}} \cdots \int_0^{x_2} dx_{n-1}\, dx_{n-2} \ldots dx_1}$$

$$= \frac{l^p (n-1)!\, p!}{(p+n-1)!} = \frac{l^p [\Gamma(n),\ \Gamma(p+1)]}{\Gamma(p+n)}.$$

Also solved by G. B. M. Zerr.

186. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

An urn contains $n=100$ balls; $a=25$ balls are stamped, at random, with the letter A; $b=30$ balls are stamped, at random, with the letter B; $c=40$ balls are stamped, at random, with the letter C; $d=50$ balls are stamped, at random, with the letter D. One ball is drawn at random; find the chance it has on it no letter, the letter A, or B, or C, or D, or the letters AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, or $ABCD$.

Solution by J. E. SANDERS, Reinersville, O., and the PROPOSER.

$(a/n)=\frac{25}{100}=\frac{1}{4}$, $(1-a/n)=\frac{3}{4}$; $(b/n)=\frac{30}{100}=\frac{3}{10}$, $(1-b/n)=\frac{7}{10}$; $(c/n)=\frac{40}{100}=\frac{2}{5}$, $(1-c/n)=\frac{3}{5}$, $(d/n)=(1-d/n)=\frac{50}{100}=\frac{1}{2}$.

$\therefore$ The required chances are in order:

$(1-a/n)(1-b/n)(1-c/n)(1-d/n)=\frac{3}{4}\cdot\frac{7}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{63}{400}$.
$(a/n)(1-b/n)(1-c/n)(1-d/n)=\frac{1}{4}\cdot\frac{7}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{21}{400}$.
$(1-a/n)(b/n)(1-c/n)(1-d/n)=\frac{3}{4}\cdot\frac{3}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{27}{400}$.
$(1-a/n)(1-b/n)(c/n)(1-d/n)=\frac{3}{4}\cdot\frac{7}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{42}{400}$.
$(1-a/n)(1-b/n)(1-c/n)(d/n)=\frac{3}{4}\cdot\frac{7}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{63}{400}$.
$(a/n)(b/n)(1-c/n)(1-d/n)=\frac{1}{4}\cdot\frac{3}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{4}{400}$.
$(a/n)(1-b/n)(c/n)(1-d/n)=\frac{1}{4}\cdot\frac{7}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{14}{400}$.
$(a/n)(1-b/n)(1-c/n)(d/n)=\frac{1}{4}\cdot\frac{7}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{21}{400}$.
$(1-a/n)(b/n)(c/n)(1-d/n)=\frac{3}{4}\cdot\frac{3}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{18}{400}$.
$(1-a/n)(b/n)(1-c/n)(d/n)=\frac{3}{4}\cdot\frac{3}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{27}{400}$.
$(1-a/n)(1-b/n)(c/n)(d/n)=\frac{3}{4}\cdot\frac{7}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{42}{400}$.
$(a/n)(b/n)(c/n)(1-d/n)=\frac{1}{4}\cdot\frac{3}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{6}{400}$.
$(a/n)(b/n)(1-c/n)(d/n)=\frac{1}{4}\cdot\frac{3}{10}\cdot\frac{3}{5}\cdot\frac{1}{2}=\frac{9}{400}$.
$(a/n)(1-b/n)(c/n)(d/n)=\frac{1}{4}\cdot\frac{7}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{14}{400}$.
$(1-a/n)(b/n)(c/n)(d/n)=\frac{3}{4}\cdot\frac{3}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{18}{400}$.
$(a/n)(b/n)(c/n)(d/n)=\frac{1}{4}\cdot\frac{3}{10}\cdot\frac{2}{5}\cdot\frac{1}{2}=\frac{6}{400}$.

187. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

Solution by the PROPOSER.

Let x and y be the coordinates of one of the points, and θ=the angle one of the lines makes with the side taken as the Y-axis. Then the length of the line from the point to the side taken as the X-axis is $y\sec\theta$.

It is only necessary to find the average length of this line, for the average length of lines drawn to the other sides is evidently the same.

When the line passes through the origin, $y\tan\theta=x$. Hence the limits of θ are 0 and $\tan^{-1}(x/y)$. Hence, the required average is,

$$A=\int_0^a\int_0^a\int_0^{\tan^{-1}(x/y)} y\sec\theta\,dx\,dy\,d\theta\div\int_0^a\int_0^a\int_0^{\tan^{-1}(x/y)} dx\,dy\,d\theta$$

$$=\int_0^a\int_0^a \log\left(\frac{x}{y}+\frac{\sqrt{(x^2+y^2)}}{y}\right)dxy\,dy\div\int_0^a\int_0^a \tan^{-1}\left(\frac{x}{y}\right)dx\,dy$$

$$=\int_0^a\left[\frac{a^2}{2}\log\left(\frac{x+\sqrt{(a^2+x^2)}}{a}\right)+\frac{x}{2}\sqrt{(a^2+x^2)}-\frac{x^2}{2}\right]dx$$

$$\div\int_0^a\left[a\tan^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\log\left(\frac{a^2+x^2}{x^2}\right)\right]dx=\frac{2a}{\pi}[\log(1+\sqrt{2})-\tfrac{1}{3}(\sqrt{2}-1)]$$

$$=.4095a.$$

Also solved by G. B. M. Zerr, whose solution will appear in the next issue.

PROBLEMS FOR SOLUTION.

ALGEBRA.

287. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

For what fraction of a year will there be the greatest difference between the interest as computed by the ordinary commercial rule and that computed by the rule of compound interest?

288. Proposed by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

Evaluate the determinant which arises in finding the inverse of the transformation, with binomial coefficients,

$$T: \quad \xi_i = \sum_{j=i}^{g-1} \binom{j}{i} x_j \quad (i=0, 1, \ldots, g-1).$$

GEOMETRY.

320. Proposed by S. F. NORRIS, Baltimore City College.

Lines are drawn from a fixed point P_1, meeting a fixed circle in P_2. On P_1P_2 a point P is taken so that $P_1P \times P_1P_2 = k^2$. Find the locus of P. Solve by analytic methods, using rectangular coordinates, and putting the result in the form,

$$(x_1^2+y_1^2-r^2)[(x-x_1)^2+(y-y_1)^2]+2k^2(x_1x+y_1y-x_1^2-y_1^2)+k^4=0.$$

321. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Prove by plane geometry the following interesting theorem:

If from a point in the plane of a triangle perpendiculars are demitted upon the three sides of the triangle, and if the area of the triangle formed by connecting the feet of these perpendiculars is denoted by $\triangle'$, the distance of the assumed point from the center of the circle circumscribed about the original triangle by R', the radius of the circumscribed circle by R, and the area of the pedal triangle by $\triangle$, then will $\triangle'/\triangle = \pm[(R^2-R'^2)/R^2]$.

CALCULUS.

242. Proposed by J. H. MEYER, S. J., Augusta, Ga.

A given sphere is to be formed into a solid composed of two equal cones on opposite sides of a common base, in such a manner that its surface may be the least possible. Find the dimensions of the solid, and compare its surface with that of the sphere.

243. Proposed by R. D. CARMICHAEL, Anniston, Ala.

The usual method for the solution of a differential equation in the form (see Cohen, *Differential Equations*, p. 22)

$$x^r y^s(my\,dx+nx\,dy)+x^\rho y^\sigma(\nu y\,dx+\nu x\,dy)=0$$

fails when (1) $n=am$, (2) $\nu=a\mu$, (3) $s-\sigma \neq a(r-\rho)$. Find the solution when the relations (1) and (2) hold. (Note that the solution desired does not depend on (3).)

244. Proposed by G. B. M. ZERR, Ph. D., Professor of Mathematics in Central Manual Training School, Philadelphia, Pa.

Fine the volume common to the solids bounded by the surfaces $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$ and $x^{1/3}z^{2/3}=(a^{1/3}-x^{1/3})(x^{2/3}+y^{2/3})$.

MECHANICS.

205. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given two points A and B not in the same horizontal nor in the same vertical line; to find the path from A to B along which a particle will slide from rest under the force of gravity alone so that the average velocity along the curve shall be a maximum.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A rigid square $ABDC$ made by smooth wires is fixed with A vertically above D. Two small equal spherical elastic beads slide down BD, CD, starting simultaneously from B and C. Find the ratio of their velocities of approach and separation at D, and how far they will separate after impact.

NOTES AND NEWS.

The eminent mathematician, Yoshio Mikami, is translating Dr. Halsted's *Rational Geometry* for publication in Japan. F.

Dr. G. B. M. Zerr has been appointed Professor of Mathematics in the Central Manual Training School of Philadelphia, at a salary of $2000 per year. F.

Professor G. W. Greenwood has resigned his position as Professor of Mathematics in Roanoke College and gone into business in Dunbar, Pa. He will, however, continue his contributions to the MONTHLY. F.

Miss Hazel Anderson, who has just received the Master's degree at the University of Chicago, will be Instructor in Mathematics at the Manual Training High School, Indianapolis, Ind. S.

Mr. G. R. Clements, who took the Master's degree at the University of Chicago in August, has been appointed to an Instructorship in Mathematics at Williams College, Williamstown, Mass. S.

Miss Mary E. Wells, who received the Master's degree at the University of Chicago in June, returns to an Instructorship in Mathematics at Mt. Holyoke College, where she graduated in 1906. S.

Dr. R. L. Boerger, formerly instructor at the University of Missouri, has been appointed to an Instructorship in Mathematics at the University of Illinois. Mr. Boerger received the Doctor's degree at the University of Chicago in August, 1907. S.

Mr. John W. Mitchell has been appointed Instructor in Matematics at the Agricultural and Mechanical College of Texas. S.

Dr. Louis Ingold returns to the University of Missouri as Instructor in Mathematics, after a two years' leave of absence, during which time he has been Fellow in Mathematics at the University of Chicago, where he has just received the Doctor's degree. S.

Dr. W. H. Bussey, who for two years has been instructor in Barnard College, Columbia University, has been elected to an Assistant Professorship in Mathematics at the University of Minnesota. Mr. Bussey took his Doctorate at the University of Chicago in 1906. S.

Dr. N. J. Lennes has been elected to an Instructorship in Mathematics at the Massachusetts Institute of Technology. Mr. Lennes has been for some years teaching mathematics in the Chicago high schools, and has just taken his Doctorate at the University of Chicago. S.

Dr. F. W. Owens has been made Instructor in Mathematics at Cornell University. For two years Mr. Owens has been teaching mathematics at the Academy of North Western University at Evanston, Ill. He took his Doctorate at the University of Chicago on August 30, 1907. S.

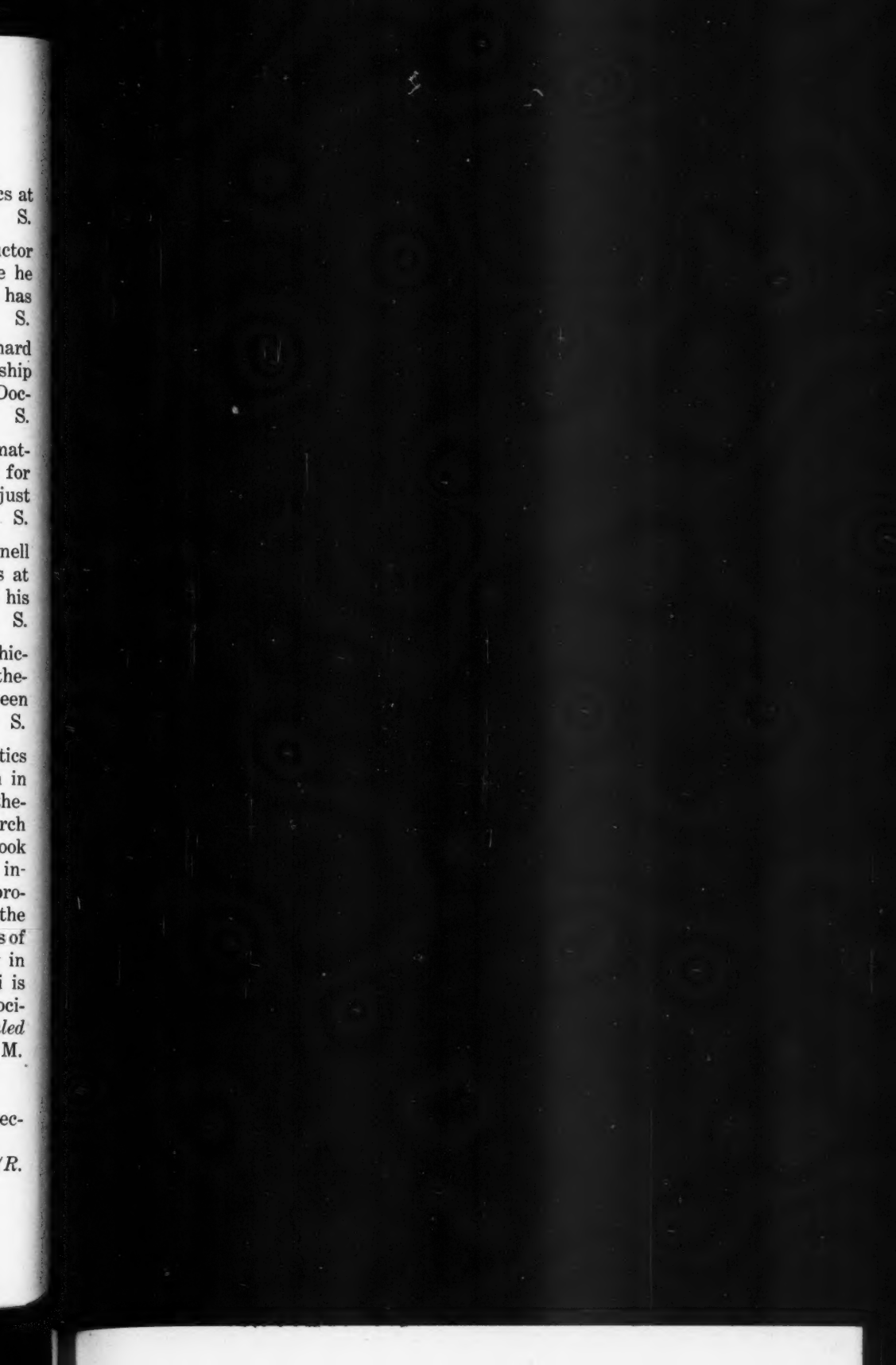
Dr. N. R. Wilson, who received his degree at the University of Chicago at the August Convocation, returns as Associate Professor of Mathematics at the University of Manitoba, Winnipeg, Manitoba. He has been there several years and was recently promoted to Associate Professor. S.

E. J. Wilczynski, Ph. D. (Berlin), Associate Professor of Mathematics in the University of California, has been appointed to a similar position in the University of Illinois. Professor Wilczynski is one of the best mathematicians of America. He has been both research assistant and research associate of the Carnegie Institute of Washington, and the 1904 Year Book contains the following about his work: "The general character of these investigations places them at the beginning of a new kind of geometry, a projective geometry which does not confine itself to the consideration of the algebraic cases, as has hitherto been the case, but which proves theorems of a more general nature by the use of differential equations, resembling in that respect the general theory of surfaces." Professor Wilczynski is chairman of the San Francisco Section of the American Mathematical Society, and is the author of *Projective Differential Geometry of Curves and Ruled Surfaces*, recently published by B. G. Teubner of Germany. M.

ERRATA.

In solution of problem 197, Mechanics, page 107, the following corrections should be noted:

Line 7, for v_1 read v_2; line 8, for v_2 read v_1; line 9, for v_1^2/r read v_2^2/R.



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NOTES ON THE GEOMETRICAL REPRESENTATION OF THE ROOTS OF EQUATIONS.

By DR. O. E. GLENN, The University of Pennsylvania.

The importance, from the historical standpoint, of the method of representing the roots of equations by means of intersecting conics, suggests that this species of graphical representation might deserve a more prominent place in teaching. Constructions of this kind were known to the Greek geometers, an account of whose discoveries may be found in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*, Chap. 11, pp. 240-243. An account of the work in the same direction by Descartes, Van Schooten and others is given in M. Cantor's *Geschichte der Mathematik*, Vol. 2, Chap. LXXVII, pp. 736-737, and a short reference to the methods in Klein's **Vortraege über Ausgewahlte Fragen der Elementargeometrie.*

In this paper constructions for the cubic, biquadratic, and quintic are obtained in what is believed to be their most practicable form. These results are then employed to derive elementary analytical properties of the equations directly from the geometrical properties of the constructions.

§ 1. The Cubic.

The roots of the general cubic

$$x^3+px+q=0 \tag{1}$$

may be obtained by eliminating z from the simultaneous pair

$$x^2+qz+p=0,$$
$$x+qz^2+pz=0,$$

x, z being independent variables. Also the same trio of x values will be obtained from this pair altered by the transformation

$$z=\frac{-y}{q}-\frac{p}{q}.$$

*See Translation by Beman and Smith.